数理解析研究所講究録
第 1936 巻 2015 年 86-94

HYPERELLIPTIC GOLDMAN LIE ALGEBRA AND ITS ABELIANIZATION

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ABSTRACT. In this article, we focus on the definition of the Goldman Lie algebra of an orbifold and its properties without fear of going off what is written in the title. The Goldman Lie algebra is a Lie algebra structure on the free vector space which spanned by the set of all free homotopy classes of oriented loops on a surface. In other words, the vector space is spanned by the set of all conjugacy classes of the fundamental group. We extend the definition of the Goldman Lie algebra of a surface to an orbifold. We consider the surface obtained by removing all singular points from an orbifold and define an ideal of the Goldman Lie algebra of this surface. The Goldman Lie algebra of the orbifold is quotient Lie algebra modulo the ideal. The Goldman Lie algebra of a orbifold is naturally linear isomorphic to the free vector space spanned by the set of all conjugacy classes of the orbifold fundamental group of it.

1. INTRODUCTION

Goldman introduced a Lie algebra, the Goldman Lie algebra, in his paper [3] on the geometry of a symplectic structure on the representation space of a surface group. The Lie algebra is defined for any oriented surface and this Lie algebra structure only depends on the homeomorphism type of the surface. The Goldman Lie algebra $\mathbb{Q}\hat{\pi}(S)$ is defined as the following. Let S be a connected oriented surface and $\hat{\pi}(S)$ the set of all free homotopy classes of oriented loops on S. $\mathbb{Q}\hat{\pi}(S)$ is the $\mathbb{Q}$-vector space on $\hat{\pi}(S)$. Let a and b be immersed loops on S such that all intersection points of $a\cup b$ are transverse double points (we call such immersed loops *generic*). Then we define the Goldman bracket of these loops by

$$[a,b]=\sum_{p\in a\cap b}\varepsilon(p;a,b)\,|a_pb_p|\in\mathbb{Q}\hat{\pi}(S),$$

where $\varepsilon(p;a,b)$ is the local intersection number of a and b at p, a_p and b_p are closed paths based at p obtained from a and b respectively. We consider these paths as elements in $\pi_1(S,p)$. $|a_pb_p|$ is the free homotopy class of an oriented loop obtained by forgetting the base point of a_pb_p.

Theorem 1.1 (Goldman [3]). *The Goldman bracket* $[\ ,\ ]$ *is well-defined on* $\hat{\pi}(S)$ *and the linear extension of this bracket defines the Lie algebra structure on* $\mathbb{Q}\hat{\pi}(S)$.

Goldman also introduced a homological version of the Goldman Lie algebra. Let $\mathbb{Q}H(S)$ be the $\mathbb{Q}$-vector space spanned by $H(S)=H_1(S,\mathbb{Z})$. When we consider X in $H(S)$ as an element of the basis of $\mathbb{Q}H(S)$, we denote it by $\langle X\rangle$. Then the bracket of $\mathbb{Q}H(S)$ is defined by

$$[\langle X\rangle,\langle Y\rangle]=\mu(X\cdot Y)\langle X+Y\rangle\in\mathbb{Q}H(S)$$

for any X and Y in $H(S)$ where μ is the intersection form on $H(S)$. The linear extension of this bracket defines a Lie algebra structure on $\mathbb{Q}H(S)$ and we call it the homological

Goldman Lie algebra of a surface S. There is a canonical surjective homomorphism from the Goldman Lie algebra to the homological Goldman Lie algebra of the same surface. This surjective Lie algebra homomorphism $\mathrm{Ab}_*\colon \mathbb{Q}\hat{\pi}(S) \to \mathbb{Q}H(S)$ is induced by the abelianization $\mathrm{Ab}\colon \pi_1(S) \to H_1(S)$.

In this article, We define the Goldman Lie algebra of a 2-orbifold by the quotient of the Goldman Lie algebra of the surface obtained from removing all singular points in Seciton 2. We remark that Chas and Gadgil [2] give another definition of the Goldman bracket for 2-orbifolds by the use of orbifold homotopies and show the Jacobi identity by a method of hyperbolic geometry. We also review its definition in this section. We will also show that underlying vector space of our Lie algebra of an orbifold is linear isomorphic to the vector space spanned by the all conjugacy classes of its orbifold fundamental group. In section 3, we describe a relationship between Goldman Lie algebras of finite Galois coverings. More precisely, the action of the covering transformation group Γ of a finite unbranched Galois covering $\tilde{S} \to S$ on the total space $\tilde{S}$ induces an action of Γ on $\mathbb{Q}\hat{\pi}(\tilde{S})$. Then the Γ-invariant part $\mathbb{Q}\hat{\pi}(\tilde{S})^{\Gamma}$ is embedded in the Goldman Lie algebra $\mathbb{Q}\hat{\pi}(S)$ of the base space. In Section 4, we give some observation of a property of Goldman Lie algebras of finite branched Galois coverings.

2. Goldman Lie algebra of orbifold

A orbifold was defined by Satake[6] (under the name of "V-manifold") and Thurston[7]. We briefly review the definition of an orbifold based on Bonahon-Siebenmann[1] and Matsumoto-Montesinos[5]. A n-dimensional orbifold is a paracompact Hausdorff space X with an orbifold atlas of folding charts $\{(\tilde{U}_i, G_i, \varphi_i, U_i)\}_{i\in I}$. $\tilde{U}_i$ is a connected smooth n-dimensional manifold, G_i a finite group acting smoothly and effectively on $\tilde{U}_i$, U_i a connected open set of X such that $X = \cup_{i\in I} U_i$, a folding map $\varphi_i\colon \tilde{U}_i \to U_i$ a continuous map which naturally induces a homeomorphism between $\tilde{U}_i/G_i$ and U_i. Furthermore, the folding charts satisfy the following compatibility condition.

- If $U_i \cap U_j \neq \emptyset$ and $\varphi_i(x) = \varphi_j(y) = p \in U_i \cap U_j$, then there exists a diffeomorphism $\psi\colon \tilde{V}_x \to \tilde{V}_y$ from an open neighborhood of $x \in \tilde{V}_x \subset \tilde{U}_i$ to an open neighborhood of $y \in \tilde{V}_y \subset \tilde{U}_j$ such that $\varphi_j\psi = \varphi_i$ and $\psi(x) = y$.

For any folding chart $(\tilde{U}_i, G_i, \varphi_i, U_i)$ and x in $\tilde{U}$, the stabilizer subgroup $(G_i)_x$ of x is the set of all elements in G_i which fixing x. The isomorphism class of $(G_i)_x$ depends only on the point $p = \varphi_i(x)$ in X. We call this isomorphism class the isotropy group of p. We define the singular set ΣX of X as the set of all points in X which have nontrivial isotropy groups. A singular point of X is an element in ΣX.

In this article, we only consider a 2-dimensional orbifold with the underlying space S which has 0-dimensional isolated singular points. We take an orbifold atlas of S as the following. $S - \Sigma S$ is a connected oriented smooth surface which denoted by S_*. A singular point p in ΣS has a folding chart $(\mathbb{D}, C_p, \varphi_p, U_p)$. The folding map $\varphi_p : \mathbb{D} \to U_p$ is a continuous map with $\varphi_p(0) = p$ where $\mathbb{D}$ is the open unit disk in $\mathbb{C}$ and U_p an open neighborhood of p. The cyclic group C_p of order m_p acts on $\mathbb{D}$ by $2\pi/m_p$-rotation. We denote the orbifold by $(S, \Sigma S)$ for the sake of simplicity. If all isotropy groups are trivial, that is $\Sigma S = \emptyset$, then S is an 2-dimensional manifold. Let p_0 be a base point in S_*. Fix a reference path u_p from p_0 to $\varphi_p(1)$ for each p in P. Let ξ be a closed path in $\mathbb{D}$ defined by Figure1. We define a closed path $\xi_p = u_p(\varphi_p \circ \xi)\bar{u}_p$ by connecting the above paths where $\bar{u}_p$ is the inverse path of u_p. This path is m_p-th power of a "meridian" of p in ΣS. $\langle \xi \rangle_S$ denotes the normal subgroup generated by a subset $\{\xi_p \mid p \in \Sigma S\}$ of $\pi_1(S_*, p_0)$ and

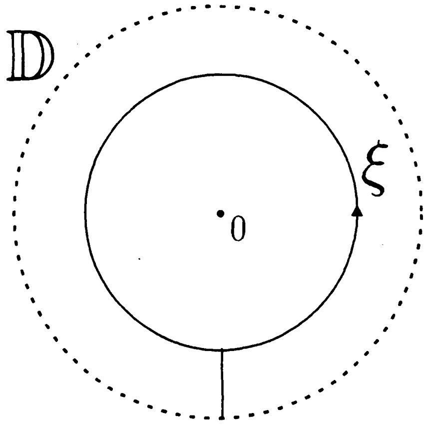

FIGURE 1. ξ

we call it the characteristic subgroup of $\pi_1(S_*, p_0)$. The orbifold fundamental group of $(S, \Sigma S)$ can be defined by $\pi_1(S_*, p_0)/\langle\xi\rangle_S$ and we denote it by $\pi_1^{\mathrm{orb}}(S, p_0)$.

We define the Goldman Lie algebra of the above orbifold $(S, \Sigma S)$ as a quotient Lie algebra of $\mathbb{Q}\hat{\pi}(S_*)$. $I(S, \Sigma S)$ denotes the vector subspace of $\mathbb{Q}\hat{\pi}(S_*)$ generated by the set

$$\{|\ell| - |\ell\xi_p| \mid \ell \in \pi_1(S_*, p_0), p \in \Sigma S\}$$

as a vector space. We remark that the definition of $I(S, \Sigma S)$ is independent of choice of reference paths. Because, If we take a new reference path v_p, then

$$\begin{aligned}|\ell| - |\ell v_p(\varphi_p \circ \xi)\bar{v}_p| &= |\ell| - |\ell(v_p\bar{u}_p)u_p(\varphi_p \circ \xi)\bar{u}_p(u_p\bar{v}_p)| \\ &= |(u_p\bar{v}_p)\ell(v_p\bar{u}_p)| - |(u_p\bar{v}_p)\ell(v_p\bar{u}_p)u_p(\varphi_p \circ \xi)\bar{u}_p| \\ &= |\ell'| - |\ell' u_p(\varphi_p \circ \xi)\bar{u}_p|\end{aligned}$$

where ℓ' is $(u_p\bar{v}_p)\ell(v_p\bar{u}_p)$ in $\pi_1(S_*, p_0)$.

Lemma 2.1. *$I(S, \Sigma S)$ is an ideal of $\mathbb{Q}\hat{\pi}(S_*)$.*

Proof. We show that $[|a|, |\ell| - |\ell\xi_p|]$ is an element in I_P for any a in $\pi_1(S_*)$. We take a representative of $|a|$ which intersects $\ell\xi_p$ transversally. If the representative intersects at points on ξ_p, then we replace it by a homotopy to keeping away from a neighborhood of $U_p \cup u_p$. We denote the representative by a'. We remark that the representative has no intersection points with ξ_p. Therefore we obtain the following equality.

$$\begin{aligned}[|a|, |\ell| - |\ell\xi_p|] &= [|a'|, |\ell| - |\ell\xi_p|] \\ &= \sum_{q \in a' \cap \ell} \varepsilon(q; a', \ell)\, |a'_q \ell_q| - \sum_{q \in a' \cap \ell\xi_p} \varepsilon(q; a', \ell\xi_p)\, |a'_q(\ell\xi_p)_q| \\ &= \sum_{q \in a' \cap \ell} \varepsilon(q; a', \ell)(|a'_q \ell_q| - |a'_q(\ell\xi_p)_q|) \\ &= \sum_{q \in a' \cap \ell} \varepsilon(q; a', \ell)(|\ell_{p_0 q} a'_q \ell_q \bar{\ell}_{p_0 q}| - |(\ell_{p_0 q} a'_q \ell_q \bar{\ell}_{p_0 q})\xi_p|) \in I_P\end{aligned}$$

where $\ell_{p_0 q}$ is the path from p_0 to q obtained by restricting ℓ to $[0, \ell^{-1}(q)]$ and $\bar{\ell}_{p_0 q}$ is its inverse path. □

Definition 2.2. *The Goldman Lie algebra of $(S, \Sigma S)$ is defined by $\mathbb{Q}\hat{\pi}(S_*)/I(S, \Sigma S)$. We denote it by $\mathbb{Q}\hat{\pi}(S, \Sigma S)$.*

Remark 2.3. *Let S be a 2-dimensional manifold with a set of specified points $P \subset S$. We define ξ_p as a meridian of p in P and the characteristic subgroup $\langle\zeta\rangle_S$ of $\pi_1(S-P)$ by considering $p \in P$ as a "order 1 singular point". Then $\mathbb{Q}\hat{\pi}(S,P) = \mathbb{Q}\hat{\pi}(S-P)/\langle\zeta\rangle_S$ is isomorphic to the Goldman Lie algebra of S.*

Let G be a group and $\hat{G}$ the set of conjugacy classes of G. $\hat{g}$ denote the conjugacy class represented by g in G. $\mathbb{Q}\hat{G}$ is the $\mathbb{Q}$-vector space spanned by $\hat{G}$. The Goldman Lie algebra of a surface S can be considered as $\mathbb{Q}\widehat{\pi_1}(S,p_0)$ by a natural bijection between $\widehat{\pi_1}(S,p_0)$ and $\hat{\pi}(S)$. The natural bijection is obtained by regarding $|a|$ as $\hat{a}$ for any a in $\pi_1(S)$.

Proposition 2.4. *There is a natural isomorphism between $\mathbb{Q}\hat{\pi}(S,\Sigma S)$ and $\mathbb{Q}\widehat{\pi_1^{orb}}(S,p_0)$.*

Proof. We denote $\pi_1^{\mathrm{orb}}(S,p_0)$ by Γ. The natural quotient map $\Phi_\#\colon \pi_1(S_*) \to \Gamma$ induces a linear map $\Phi_*\colon \mathbb{Q}\hat{\pi}(S_*) \to \mathbb{Q}\hat{\Gamma}$. The kernel of Φ_* is generated by a subset $\{|a|-|b| \mid |\Phi_\#(a)| = |\Phi_\#(b)| \text{ and } a,b \in \pi_1(S_*)\}$ of $\mathbb{Q}\hat{\pi}(S_*)$. The condition $|\Phi_\#(a)| = |\Phi_\#(b)|$ means that $\Phi_\#(a)$ is conjugate to $\Phi_\#(b)$ in Γ. Therefore $a^{-1}gbg^{-1}$ is contained in $\langle\xi\rangle_S$ for some g in $\pi_1(S_*)$. We can denote $a^{-1}gbg^{-1}$ by $\prod_{i=1}^n h_i\xi_{p_i}h_i^{-1}$ where h_i is in $\pi_1(S_*)$ and p_i in ΣS for each i. Then the generator

$$\begin{aligned}
|a|-|b| &= |a| - \left|a\prod_{i=1}^n h_i\xi_{p_i}h_i^{-1}\right| \\
&= |a| - |ah_1\xi_{p_1}h_1^{-1}| + \sum_{i=1}^{n-1}\left(\left|a\prod_{j=1}^{i} h_j\xi_{p_j}h_j^{-1}\right| - \left|a\prod_{j=1}^{i+1} h_j\xi_{p_j}h_j^{-1}\right|\right) \\
&= \sum_{i=1}^n |a_i| - |a_i\xi_{p_i}|
\end{aligned}$$

where $a_1 = h_1^{-1}ah_1$ and a_i is defined inductively by $h_i^{-1}a_{i-1}h_i$. Consequently, The ideal $I(S,\Sigma S)$ includes the kernel of Φ_*. The reverse inclusion is clear. Therefore Φ_* induce an isomorphism $\Phi\colon \mathbb{Q}\hat{\pi}(S,\Sigma S) \to \mathbb{Q}\hat{\Gamma}$. □

Chas and Gadgil[2] give a group-theoretic definition of the Goldman bracket for an orbifold by using hyperbolic geometry. We will review this definition. Let Γ be a discrete subgroup of the orientation preserving isometric group of the upper half plane $\mathbb{H}$ with the hyperbolic metric and base point $\tilde{p}_0$. For any x and y in Γ, $I(x,y)$ denote the empty set if x or y are non-hyperbolic elements, otherwise,

$$I(x,y) = \{\, XgY \in X\backslash\Gamma/Y \mid \mathrm{Ax}(x) \cap g\,\mathrm{Ax}(y) \neq \emptyset, g \in \Gamma \,\}$$

where X and Y are infinite cyclic subgroups generated by x and y respectively, $X\backslash\Gamma/Y$ is the set of double coset of X and Y in Γ. The Goldman bracket $[\ ,\ ]_\Gamma$ on $\mathbb{Q}\hat{\Gamma}$ is given by

$$[\hat{x},\hat{y}]_\Gamma = \sum_{XgY \in I(x,y)} \varepsilon(x, gyg^{-1})\widehat{xgyg^{-1}}$$

for any x and y in Γ. $\varepsilon(x,gyg^{-1})$ is the algebraic intersection number of $\mathrm{Ax}(x)$ and $g\,\mathrm{Ax}(y)$. If x is a hyperbolic element in Γ, we can uniquely determine the geodesic line $\mathrm{Ax}(x)$ in $(\mathbb{H}_0,\tilde{p}_0)$ which fixed by the action of x. We call the geodesic line $\mathrm{Ax}(x)$ the axis of x. The orientation of the axis of a hyperbolic element is defined by the direction from the repelling point to the attractive point.

We will observe a relationship between our bracket and the above bracket. The orbit space $S_\Gamma = \mathbb{H}/\Gamma$ gives an orbifold. We denote the projection by $\varpi\colon \mathbb{H} \to S_\Gamma$ and

$\mathbb{H}-\varpi^{-1}(\Sigma S_\Gamma)$ by $\mathbb{H}_0$. We take a base point $\tilde{p}_0$ in $\mathbb{H}_0$ and $p_0=\varpi(\tilde{p}_0)$. The orbifold fundamental group $\pi_1^{\mathrm{orb}}(S_\Gamma,p_0)$ is isomorphic to Γ. The characteristic subgroup of $\pi_1((S_\Gamma)_*,p_0)$ is equal to $\varpi_\#(\pi_1(\mathbb{H}_0,\tilde{p}_0))$. (Refer to [4], [5], [7] etc.) Proposition2.4 gives a linear isomorphism $\Phi\colon \mathbb{Q}\hat{\pi}(S_\Gamma,\Sigma S_\Gamma)\to\mathbb{Q}\hat{\Gamma}$. Therefore $\mathbb{Q}\hat{\Gamma}$ has another Lie algebra structure induced from $\mathbb{Q}\hat{\pi}(S_\Gamma,\Sigma S_\Gamma)$.

Question 2.5. *Does Φ induce a Lie algebra isomorphism from $(\mathbb{Q}\hat{\pi}(S_\Gamma,\Sigma S_\Gamma),[\ ,\ ])$ to $(\mathbb{Q}\hat{\Gamma},[\ ,\ ]_\Gamma)$?*

Observation of Question2.5. Let x be an element in Γ and $\tilde{p}$ in $\mathbb{H}_0$ a starting point. We take a path $\gamma_x^{\tilde{p}}$ from $\tilde{p}$ to $x\tilde{p}$ in $\mathbb{H}_0$. $|\varpi\gamma_x^{\tilde{p}}|$ defines a unique element independent of a choice of starting points in $\mathbb{H}_0$. The inverse of Φ sends $\hat{x}$ to $|\varpi\gamma_x^{\tilde{p}}|$. If x be an elliptic or parabolic element in Γ, then $\varpi\gamma_x^{\tilde{p}}$ is freely homotopic to a neighborhood of a singular point or a puncture of S_Γ. These elements are contained in the center of $\mathbb{Q}\hat{\pi}(S_\Gamma,\Sigma S_\Gamma)$. Let x and y be hyperbolic elements in Γ and $\tilde{p}$ a point on $\mathrm{Ax}(x)\cap\mathbb{H}_0$ outside of $\cup_{g\in\Gamma} g\,\mathrm{Ax}(y)$. We denote the geodesic half line from $\tilde{p}$ to $x\tilde{p}$ in $\mathrm{Ax}(x)$ by $\gamma'^{\tilde{p}}_x$. ($\gamma'^{\tilde{p}}_x$ does not contain $x\tilde{p}$) There is a bijection between the set of all intersection points (counting multiplicities) of $\varpi\gamma'^{\tilde{p}}_x$ and $\varpi\gamma'^{\tilde{p}'}_y$ and $\gamma'^{\tilde{p}}_x\cap(\cup_{g\in\Gamma}(g\,\mathrm{Ax}(y)))$ for some $\tilde{p}'$ on $\mathrm{Ax}(y)$. Furthermore, There is a bijection between $\gamma'^{\tilde{p}}_x\cap(\cup_{g\in\Gamma}(g\,\mathrm{Ax}(y)))$ and $I(x,y)$. (Refer to Chas and Gadgil[2]) If XgY is in $I(x,y)$, then $\gamma'^{\tilde{p}}_x$ and $g\,\mathrm{Ax}(y)$ have a unique intersection point $\tilde{q}$ in $\mathbb{H}$. We can obtain a piecewise geodesic path $\gamma'^{\tilde{q}}_x\gamma'^{x\tilde{q}}_{xgyg^{-1}x^{-1}}$ from $\tilde{q}$ to $xgyg^{-1}\tilde{q}$. We remark that $x\tilde{q}$ is on $xg\,\mathrm{Ax}(y)$ which is the axis of $xgyg^{-1}x^{-1}$. $\varpi\gamma'^{\tilde{q}}_x$ and $\varpi\gamma'^{x\tilde{q}}_{xgyg^{-1}x^{-1}}$ represent $\Phi^{-1}(x)$ and $\Phi^{-1}(y)$ respectively. These representatives intersect at $q=\varpi(\tilde{q})$. The projection $\varpi(\gamma'^{\tilde{q}}_x\gamma'^{x\tilde{q}}_{xgyg^{-1}x^{-1}})$ is a product of based loops $(\varpi\gamma'^{\tilde{q}}_x)_q$ and $(\varpi\gamma'^{x\tilde{q}}_{xgyg^{-1}x^{-1}})_q$ where $q=\varpi(\tilde{q})$. Therefore,

$$\begin{aligned}
\Phi^{-1}([\hat{x},\hat{y}]_\Gamma) &= \Phi^{-1}\Big(\sum_{XgY\in I(x,y)}\varepsilon(x,gyg^{-1})\widehat{xgyg^{-1}}\Big)\\
&= \sum_{\tilde{q}\in\gamma'^{\tilde{p}}_x\cap(\cup_{g\in\Gamma}(g\,\mathrm{Ax}(y)))}\varepsilon(q;\varpi\gamma'^{\tilde{q}}_x,\varpi\gamma'^{x\tilde{q}}_{xgyg^{-1}x^{-1}})\left|\varpi(\gamma'^{\tilde{q}}_x\gamma'^{x\tilde{q}}_{xgyg^{-1}x^{-1}})\right|\\
&= \sum_{q\in\varpi\gamma'^{\tilde{p}}_x\cap\varpi\gamma'^{\tilde{p}'}_y}\varepsilon(q;\varpi\gamma'^{\tilde{q}}_x,\varpi\gamma'^{x\tilde{q}}_{xgyg^{-1}x^{-1}})\left|(\varpi\gamma'^{\tilde{q}}_x)_q(\varpi\gamma'^{x\tilde{q}}_{xgyg^{-1}x^{-1}})_q\right|\\
&= \left[\left|\varpi\gamma'^{\tilde{q}}_x\right|,\left|\varpi\gamma'^{x\tilde{q}}_{xgyg^{-1}x^{-1}}\right|\right]\\
&= \left[\Phi^{-1}(\hat{x}),\Phi^{-1}(\widehat{xgyg^{-1}x^{-1}})\right]\\
&= \left[\Phi^{-1}(\hat{x}),\Phi^{-1}(\hat{y})\right]
\end{aligned}$$

if all intersection points are not contained in ΣS_Γ. I cannot confirm yet when q is contained in ΣS_Γ. □

Remark 2.6. *Our definition of the bracket is applicable to "bad" orbifolds. (A bad orbifold cannot be covered by a manifold* [7]*)*

We will define the homological Goldman Lie algebra of an orbifold. The first homology group of an orbifold $(S,\Sigma S)$ with coefficient in $\mathbb{Z}$ is the abelianization of the orbifold fundamental group. We denote it by $H(S,\Sigma S)$. We remark that there exists a unique homomorphism from $H_1(S_*,\mathbb{Z})$ to $H(S,\Sigma S)$ by the universal property of abelianization. Let

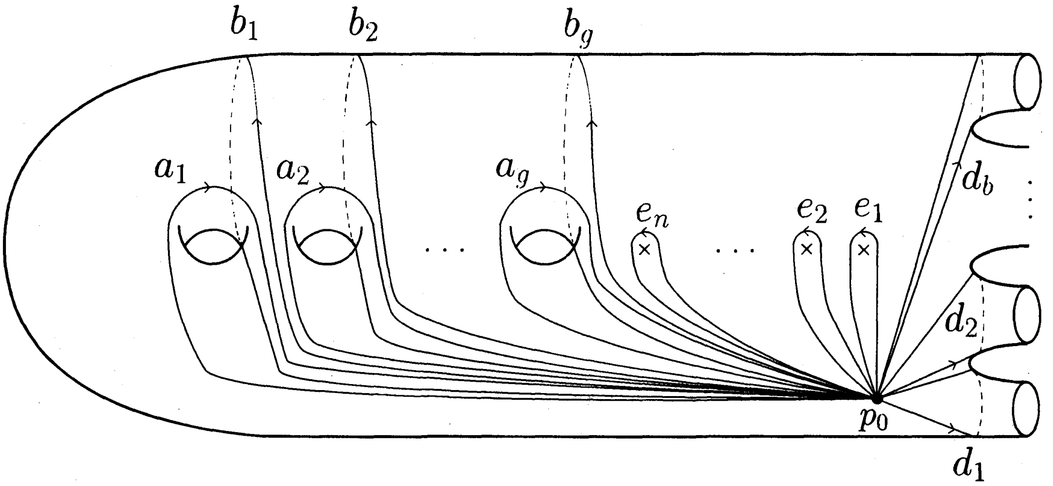

FIGURE 2. generators of $\pi_1((S_{g,b}))_*$

$(S_{g,b}, \Sigma S_{g,b})$ be an orbifold where $S_{g,b}$ is a compact connected oriented smooth surface of genus g with b boundary components. Denote $\Sigma S_{g,b}$ by $Q = \{q_1, q_2, \dots, q_n\}$ and the order of the isotropy group of q_k by m_k for each $k = 1, 2, \dots, n$. We take a generating set of $\pi_1((S_{g,b})_*, p_0)$ by $\{a_i, b_i, d_j, e_k \mid i = 1, 2, \dots, g,\ j = 1, 2, \dots, b,\ k = 1, 2, \dots, n\}$. (See figure2) We denote generators of $H_1((S_{g,b})_*, \mathbb{Z})$ by A_i, B_i, D_j and E_k which correspond to a_i, b_i, d_j and e_k respectively. Then the first homology group of $H(S, Q)$ is described by

$$H(S,Q) \cong \bigoplus_{i=1}^{g}(\mathbb{Z}A_i \oplus \mathbb{Z}B_i) \bigoplus_{j=1}^{b} \mathbb{Z}D_j \bigoplus_{k=1}^{n} \mathbb{Z}_{m_k}E_k / \langle \sum_{j=1}^{b} D_j + \sum_{k=1}^{n} E_k \rangle.$$

The intersection form μ: $H((S_{g,b})_*) \times H((S_{g,b})_*) \to \mathbb{Z}$ induces μ^{orb}: $H((S_{g,b}), Q) \times H(S_{g,b}, Q) \to \mathbb{Z}$ because μ is trivial on D_j and E_k. We can define a bracket on $\mathbb{Q}H(S_{g,b}, Q)$ by

$$[\langle X \rangle, \langle Y \rangle] = \mu^{\mathrm{orb}}(X, Y)\langle X + Y \rangle$$

for any X and Y in $H(S_{g,b}, Q)$.

Definition 2.7. *The homological Goldman Lie algebra of an orbifold $(S_{g,b}, Q)$ is the $\mathbb{Q}$-vector space $\mathbb{Q}H(S_{g,b}, Q)$ equipped with the above bracket.*

Consequently, we obtain the following commutative diagram of Lie algebras.

$$\begin{array}{ccc} \mathbb{Q}\hat{\pi}((S_{g,b})_*) & \xrightarrow{\mathrm{Ab}_*} & \mathbb{Q}H((S_{g,b})_*) \\ \downarrow & & \downarrow \\ \mathbb{Q}\hat{\pi}(S_{g,b}, Q) & \xrightarrow{\mathrm{Ab}_*} & \mathbb{Q}H(S_{g,b}, Q). \end{array}$$

3. GOLDMAN LIE ALGEBRA OF FINITE GALOIS COVERING

Let $\tilde{S}$ and S be connected oriented smooth surfaces. An orientation preserving self-diffeomorphism γ of $\tilde{S}$ induces a bijective map from $a \cap b$ to $\gamma a \cap \gamma b$ where a and b are generic immersed loops on $\tilde{S}$. The local intersection number of a and b at p agrees with that of γa and γb at $\gamma(p)$ because γ preserves orientation of $\tilde{S}$. Therefore we can obtain an automorphism γ_* of the Goldman Lie algebra $\mathbb{Q}\hat{\pi}(\tilde{S})$ given by $\gamma_*(|x|) = |\gamma x|$ for any x in $\pi_1(\tilde{S})$.

Let $f\colon \tilde{S} \to S$ be a finite Galois covering with no branched points and Γ the covering transformation group of it. We know that Γ acts on $\mathbb{Q}\hat{\pi}(\tilde{S})$ from the previous discussion. Let $\mathbb{Q}\hat{\pi}(\tilde{S})^{\Gamma}$ denote the Γ-invariant part of $\mathbb{Q}\hat{\pi}(\tilde{S})$.

Lemma 3.1. *$\mathbb{Q}\hat{\pi}(\tilde{S})^{\Gamma}$ is a Lie subalgebra of $\mathbb{Q}\hat{\pi}(\tilde{S})$.*

Proof. $\gamma_*[v,w] = [\gamma_* v, \gamma_* w] = [v,w]$ for any v and w in $\mathbb{Q}\hat{\pi}(\tilde{S})^{\Gamma}$ and γ in Γ. Therefore $\mathbb{Q}\hat{\pi}(\tilde{S})^{\Gamma}$ is a Lie subalgebra of $\mathbb{Q}\hat{\pi}(\tilde{S})$. □

We remark that the Lie subalgebra $\mathbb{Q}\hat{\pi}(\tilde{S})^{\Gamma}$ is generated by the set $\Gamma\hat{\pi}(\tilde{S}) = \left\{\Gamma(a) \mid a \in \pi_1(\tilde{S})\right\}$ where $\Gamma(a)$ denotes $\sum_{\gamma\in\Gamma} |\gamma a|$. We denote the vector subspace by $\mathbb{Q}\Gamma\hat{\pi}(\tilde{S})$. If $u = \sum_{a\in\pi_1(\tilde{S})} r_a\,|a|$ is an element in $\mathbb{Q}\hat{\pi}(\tilde{S})^{\Gamma}$, then $u = \sum_{a\in\pi_1(\tilde{S})}(r_a/\#\Gamma)\Gamma(a)$ where $\#\Gamma$ is the number of elements in Γ. Therefore the vector subspace generated by $\Gamma\hat{\pi}(\tilde{S})$ includes $\mathbb{Q}\hat{\pi}(\tilde{S})^{\Gamma}$. The reverse inclusion is obvious. Consequently, $\mathbb{Q}\hat{\pi}(\tilde{S})^{\Gamma}$ is isomorphic to $\mathbb{Q}\Gamma\hat{\pi}(\tilde{S})$. If we take the coefficient of the Goldman Lie algebra in $\mathbb{Z}$, then we cannot show the isomorphism.

We define a linear map $\hat{f}\colon \mathbb{Q}\hat{\pi}(\tilde{S})^{\Gamma} \to \mathbb{Q}\hat{\pi}(S)$ by $\hat{f}(\Gamma(a)) = |f_{\#}(a)|$ where $f_{\#}\colon \pi_1(\tilde{S}) \to \pi_1(S)$ is the injective homomorphism induced by f.

Lemma 3.2. *$\mathbb{Q}\hat{\pi}(\tilde{S})^{\Gamma}$ is the free vector space on $\Gamma\hat{\pi}(\tilde{S})$.*

Proof. If $\hat{f}(\Gamma(a)) = \hat{f}(\Gamma(b))$, that is $|f_{\#}(a)| = |f_{\#}(b)|$, hold for any a and b in $\pi_1(\tilde{S})$, then there exist γ in Γ which satisfies $|a| = |\gamma b|$. We obtain $\Gamma(a) = \Gamma(b)$. This implies that $|f_{\#}(a)|$ is not freely homotopic to $|f_{\#}(b)|$ if $\Gamma(a) \neq \Gamma(b)$. Let $s_1\Gamma(a_1) + s_2\Gamma(a_2) + \ldots + s_k\Gamma(a_k) = 0$ for all $s_1,\ldots,s_k \in \mathbb{Q}$ where $\Gamma(a_1),\ldots,\Gamma(a_k)$ are distinct elements in $\Gamma\hat{\pi}(\tilde{S})$. Then $s_1\,|f_{\#}(a_1)| + \cdots + s_k\,|f_{\#}(a_k)| = 0$ holds in $\mathbb{Q}\hat{\pi}(S)$. Therefore $s_1 = \cdots = s_k = 0$ □

Proposition 3.3. *$\hat{f}\colon \mathbb{Q}\hat{\pi}(\tilde{S})^{\Gamma} \to \mathbb{Q}\hat{\pi}(S)$ is an injective Lie algebra homomorphism.*

Proof. We can see that $\hat{f} : \Gamma\hat{\pi}(\tilde{S}) \to \hat{\pi}(S)$ is injective because of the proof of Lemma 3.2. We show that $\hat{f}$ is a Lie algebra homomorphism. For any a and b in $\pi_1(\tilde{S})$,

$$[\Gamma(a),\Gamma(b)] = \sum_{\gamma,\gamma'\in\Gamma} [|\gamma' a|, |\gamma b|] = \sum_{\gamma,\gamma'\in\Gamma} \gamma'_*\left[|a|, \left|\gamma'^{-1}\gamma b\right|\right] = \sum_{\gamma,\gamma'\in\Gamma} \gamma'_*[|a|, |\gamma b|].$$

Therefore we obtain that

$$\hat{f}([\Gamma(a),\Gamma(b)]) = \sum_{\gamma\in\Gamma} f_*([|a|, |\gamma b|])$$

where $f_*\colon \mathbb{Q}\hat{\pi}(\tilde{S}) \to \mathbb{Q}\hat{\pi}(S)$ is the induced linear map obtained from f. We take generic immersed loops $|a|$ and $|b|$ as representatives of a and b respectively. Let $H : (S^1 \sqcup S^1) \times [0,1] \to S$ be a homotopy such that

$$H(\,\cdot\,,0) = f\circ|a| \sqcup f\circ|b|\,, H(\,\cdot\,,1) = a' \sqcup b'$$

where $a' \sqcup b'$ is generic immersion. We remark that $|a| \sqcup \gamma\circ|b|$ is a lift of $f\circ|a| \sqcup f\circ|b|$ for each $\gamma \in \Gamma$. We can obtain a lift of H such that

$$\tilde{H}_\gamma(\,\cdot\,,0) = |a| \sqcup \gamma\circ|b|\,,\ \tilde{H}_\gamma(\,\cdot\,,1) = \tilde{a}' \sqcup \tilde{b}'_\gamma$$

because of the homotopy lifting property. $a' \sqcup b'$ intersects tangentially if $\tilde{a}' \sqcup \tilde{b}'_\gamma$ intersects tangentially. $a' \sqcup b'$ has triple points if $\tilde{a}' \sqcup \tilde{b}'_\gamma$ has triple points. Therefore $\tilde{a}' \sqcup \tilde{b}'_\gamma$ is generic immersion. Next, we observe a correspondence between intersection points of $a' \sqcup b'$ and $\tilde{a}' \sqcup \tilde{b}'_\gamma$. We show that there uniquely exists a lift of intersection point p of a' and b' on $\tilde{a}' \cap \tilde{b}'_\gamma$. We take a lift $\tilde{p}$ on $\tilde{a}'$ for each intersection point p of a' and b'. There is γ in Γ such that $\tilde{p}$ is on $\tilde{b}'_\gamma$ because the action of Γ is transitive. If there is another lift on $\tilde{a}'$, then p is a triple (or more multiple) point. Therefore the lift $\tilde{p}$ is uniquely determined. There is no another lift of b' such that $\tilde{p}$ is on it for the same reason. Accordingly, we obtain a bijective map between $\cup_{\gamma\in\Gamma}\tilde{a}' \cap \tilde{b}'_\gamma$ and $a' \cap b'$ by restricting f. The following equality holds because f is an orientation preserving local diffeomorphism.

$$f \circ (\tilde{a}'_{\tilde{p}}\tilde{b}'_{\gamma\tilde{p}}) = a'_p b'_p,\ \varepsilon(\tilde{p}; \tilde{a}', \tilde{b}'_\gamma) = \varepsilon(p; a', b').$$

Therefore

$$\begin{aligned}
\hat{f}[\Gamma(a), \Gamma(b)] &= \sum_{\gamma\in\Gamma} f_*([|a|, |\gamma b|]) \\
&= \sum_{\gamma\in\Gamma} f_*([|\tilde{a}'|, |\tilde{b}'_\gamma|]) \\
&= \sum_{\gamma\in\Gamma} f_*\left(\sum_{\tilde{p}\in\tilde{a}'\cap\tilde{b}'_\gamma} \varepsilon(\tilde{p}; \tilde{a}', \tilde{b}'_\gamma) |\tilde{a}'_{\tilde{p}}\tilde{b}'_{\gamma\tilde{p}}| \right) \\
&= \sum_{\tilde{p}\in\cup_{\gamma\in\Gamma}\tilde{a}'\cap\tilde{b}'_\gamma} \varepsilon(\tilde{p}; \tilde{a}', \tilde{b}'_\gamma) \left| f \circ (\tilde{a}'_{\tilde{p}}\tilde{b}'_{\gamma\tilde{p}}) \right| \\
&= \sum_{p\in a'\cap b'} \varepsilon(p; a', b') \left| a'_p b'_p \right| \\
&= [a', b'] = [|f_\#(a)|, |f_\#(b)|] = [\hat{f}(\Gamma(a)), \hat{f}(\Gamma(b))].
\end{aligned}$$

From the above, $\hat{f} : \mathbb{Q}\hat{\pi}(\tilde{S})^\Gamma \to \mathbb{Q}\hat{\pi}(S)$ is a homomorphism. □

4. Goldman Lie algebra of finite branched Galois covering

In this section, We consider a case that $f\colon \tilde{S} \to S$ is a finite Galois covering with branched points. Let $(S, \Sigma S)$ be an orbifold, $\tilde{S}$ and S_* connected oriented smooth surfaces, $f\colon \tilde{S} \to S$ a continuous surjective map such that the restriction $f_0\colon \tilde{S}_0 \to S_*$ is a finite Galois covering with the covering transformation group Γ where $\tilde{S}_0 = \tilde{S} - f^{-1}(\Sigma S)$. We denote $f^{-1}(\Sigma S)$ by $\tilde{P}$. f gives a finite uniformization of $(S, \Sigma S)$, that is an orbifold atlas is given by $\{(\tilde{U}_p, C_p, f|_{\tilde{U}_p}, U_p)\}_{p\in S}$ where U_p is a open neighborhood of p and $\tilde{U}_p$ is a connected component of $f^{-1}(U_p)$. Fix base points $\tilde{p}_0$ in $\tilde{S}_0$ and $p_0 = f(\tilde{p}_0)$ in S. Take a m_p-th power of meridian ξ_p of $p \in \Sigma S$ by the same way as Sect.2 for each p in ΣS. We also define a meridian $\zeta_{\tilde{p}}$ for each $\tilde{p} \in \tilde{P}$. (See Remark2.3) The ideal $I(\tilde{S}, \tilde{P})$ is Γ-invariant subspace of $\mathbb{Q}\hat{\pi}(\tilde{S}_0)$. We can show the following theorem.

Theorem 4.1. *There exists the injective homomorphism $\hat{f}\colon \mathbb{Q}\hat{\pi}(\tilde{S}_0)^{\Gamma}/I(\tilde{S},\tilde{P})^{\Gamma} \to \mathbb{Q}\hat{\pi}(S,P)$ satisfying the following commutative diagram.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I(\tilde{S},\tilde{P})^{\Gamma} & \longrightarrow & \mathbb{Q}\hat{\pi}(\tilde{S}_0)^{\Gamma} & \longrightarrow & \mathbb{Q}\hat{\pi}(\tilde{S}_0)^{\Gamma}/I(\tilde{S},\tilde{P})^{\Gamma} & \longrightarrow & 0 \\
 & & {\scriptstyle \hat{f}_0|_{I(\tilde{S},\tilde{P})^{\Gamma}}}\big\downarrow & & {\scriptstyle \hat{f}_0}\big\downarrow & & {\scriptstyle \hat{f}}\big\downarrow & & \\
0 & \longrightarrow & I(S,\Sigma S) & \longrightarrow & \mathbb{Q}\hat{\pi}(S_*) & \longrightarrow & \mathbb{Q}\hat{\pi}(S,\Sigma S) & \longrightarrow & 0.
\end{array}$$

Sketch of proof. The horizontal sequences are exact and $\hat{f}_0$ an injective Lie algebra homomorphism by Proposition3.3. We can prove the Theorem by carefully chase the commutative diagram. □

Remark 4.2. $\mathbb{Q}\hat{\pi}(\tilde{S}_0)^{\Gamma}/I(\tilde{S},\tilde{P})^{\Gamma}$ *is included in* $\mathbb{Q}\hat{\pi}(\tilde{S})^{\Gamma}$.

Finally, We only give a definition of the hyperelliptic Goldman Lie algebra of a surface. We denote a closed surface of genus g by S_g and fix a hyperelliptic involution ι of S_g. ι gives a degree 2 uniformization of $h\colon S_g \to (S,\Sigma S^2)$ where S^2 is the 2-sphere and all of these singular points have order 2 isotropy groups. We call the ι-invariant part of $\mathbb{Q}\hat{\pi}(S_g)$ as the *hyperelliptic Goldman Lie algebra* of S_g.

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